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# Isolated systems and integer-spin radiation fields

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**Abstract.** A certain class of integer-spin, massless fields, which appears to be sufficiently broad to include the radiation fields generated by all well behaved flat-space isolated systems, is considered. These fields satisfy an asymptotic flatness condition which is weaker than the usual peeling condition. Expressions which have the form of an integral over null infinity are obtained for their total momentum and angular momentum, and are shown to be compatible with a well defined symplectic product.

## 1. Introduction

In two-component spinor notation, a free spin-$s$ massless field may be represented by a completely symmetric spinor $\phi_{A\ldots Z}$ with $2s$ indices which satisfies the field equations $\nabla^{A}_{A'}\phi_{A\ldots Z}=0$ everywhere, where $\nabla_{AA'}$ denotes the spinor covariant derivative (Penrose 1968). For example, when $s=1$, $\phi_{AB}$ represents a free Maxwell field with field strength $F_{ab}=\varepsilon_{A'B'}\phi_{AB}+\varepsilon_{AB}\phi_{A'B'}$; when $s=2$, $\phi_{ABCD}$ represents a free linearised Weyl field with field strength

$$C_{abcd}=\varepsilon_{A'B'}\varepsilon_{C'D'}\phi_{ABCD}+\varepsilon_{AB}\varepsilon_{CD}\phi_{A'B'C'D'}.$$

Of particular importance in certain physical applications, such as scattering, are those free integer-spin, massless fields which may be used to describe outgoing radiation fields produced by well behaved isolated systems. By such a system we have in mind an isolated collection of mutually interacting, spatially bounded sources, with no incoming radiation, which exhibits typical scattering behaviour. Such fields should obviously carry finite momentum and angular momentum, and satisfy some asymptotic flatness (AF) condition. It is, however, important to choose this AF condition to be of the correct strength. If it is too strong, the resulting class of fields may be too restricted to describe a general outgoing radiation field; if it is too weak, the resulting fields may carry infinite momentum and angular momentum, and thus would not be suitable for describing the outgoing radiation field generated by a well behaved isolated system. In this paper we shall first define a set of AF conditions which appear to be of the appropriate strength for radiation fields, and then consider properties of the resulting class of fields. Apart from § 5, we shall deal exclusively with flat-space systems.

According to Penrose's conformal criterion, a field $\phi$ is said to be AF at future null infinity $\mathscr{I}^+$ if its conformally related field $\hat{\phi}=\Omega^{-1}\phi$ is well defined and smooth on $\mathscr{I}^+$ (Penrose 1968). Here $\Omega$ is the conformal factor which satisfies $\Omega=0$ and

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$\nabla_a \Omega \neq 0$ on $\mathscr{I}^+$. This implies that the field satisfies the so-called peeling conditions

$$\phi_i = \phi_i^0 r^{-2s-1+i} + \mathrm{O}(r^{-2s-1-\epsilon+i}) \qquad (\epsilon > 0) \tag{1.1}$$

where $(u, r, \zeta, \bar{\zeta})$ is any retarded, radially null coordinate system (flat-space Bondi system) with $r \sim \Omega^{-1}$. In accordance with the standard spin coefficient notation, $\phi_i$ are defined by

$$\phi_i = \phi_{A\ldots L M\ldots Z} \underbrace{o^A \ldots o^L}_{2s-i} \underbrace{\iota^M \ldots \iota^Z}_{i} \tag{1.2}$$

where $o_A o_{A'} = \nabla_{AA'} u$ and $o_A \iota^A = 1$.

Even though conditions (1.1) are satisfied by the retarded field produced by a well behaved isolated system, they can be violated by the corresponding advanced and radiation fields. To see how this can come about, consider an isolated Maxwell system consisting of two charged particles which interact via their retarded fields, and which have well defined asymptotic velocities in the sense that $v^a = v_0^a + \mathrm{O}^*(\tau^{-\epsilon})$. Due to the long-range effects of the Coulomb part of the interaction, which is dominant at asymptotic times, the velocity of either particle can be shown to have the asymptotic form

$$v^a = v_0^a + A^a \tau^{-1} + \mathrm{O}^*(\tau^{-(\epsilon+1)}) \qquad (\epsilon > 0) \tag{1.3}$$

where $\tau$ is proper time (Ludvigsen 1981a). From this behaviour it can be deduced, using the LW potentials, that the advanced field $\phi_{\mathrm{adv}}$, and hence the radiation field $\phi_{\mathrm{ret}} - \phi_{\mathrm{adv}}$, satisfy

$$\begin{aligned} &\phi_0 = \mathrm{O}^*(r^{-(2+\epsilon)}), \qquad \phi_1 = \phi_1^0 r^{-2} + \mathrm{O}^*(r^{-(2+\epsilon)}), \\ &\phi_2 = \phi_2^0 r^{-1} + \mathrm{O}^*(r^{-(1+\epsilon)}). \end{aligned} \tag{1.4}$$

Conditions (1.1) are therefore violated. (The asymptotic order symbol $\mathrm{O}^*$ used in (1.3) and (1.4) is defined as follows: $f(r, y^a) = \mathrm{O}^*(r^{-i})$ means $f(r, y^a) = \mathrm{O}(r^{-i})$ and $\Delta D^n f(ry^a) = \mathrm{O}(r^{-(i+n)})$, where $D = \partial/\partial r$, and $\Delta$ represents any combination of $y^a$ derivatives. This order symbol was first used in connection with AF fields by Penrose (1965).)

Using (1.3), one can further show that the asymptotic component $\phi_1^0$ of the radiation field satisfies

$$\phi_1^0 = \mathrm{O}^*(u^{-\epsilon}) \qquad \text{for large positive } u$$

and

$$\phi_1^0 = e_1(\overset{+}{V}{}_1^2 - \overset{-}{V}{}_1^2) + e_2(\overset{+}{V}{}_2^2 - \overset{-}{V}{}_2^2) + \mathrm{O}^*(u^{-\epsilon}) \qquad \text{for large negative } u,$$

where $e_i$ is the charge of the $i$th particle and $\overset{+(-)}{V_i}$ are real, non-zero weighting factors depending on the particle's velocity in the asymptotic future (past). Thus, for non-trivial scattering with real charges, $\lim_{u\to-\infty} \phi_1^0$ is real and non-zero.

A similar argument also holds for gravitating systems. Consider an isolated scattering event consisting of two massive particles. If we assume that at asymptotic times, when they are far apart, the particles interact via their linearised retarded gravitational fields, it is again possible to show that (1.3) holds. This can then be used to show that the resulting linearised radiation Weyl field satisfies

$$\begin{aligned} &\phi_0 = \mathrm{O}^*(r^{-(3+\epsilon)}), \qquad \phi_1 = \mathrm{O}^*(r^{-(3+\epsilon)}), \qquad \phi_2 = \phi_2^0 r^{-3} + \mathrm{O}^*(r^{-(3+\epsilon)}), \\ &\phi_3 = \phi_3^0 r^{-2} + \mathrm{O}^*(r^{-(2+\epsilon)}), \qquad \phi_4 = \phi_4^0 r^{-1} + \mathrm{O}^*(r^{-(1+\epsilon)}), \end{aligned} \tag{1.5}$$

where $\lim_{u\to-\infty} \phi_2^0$ is real. Thus conditions (1.1) are again violated.

From these two examples it is clear that the Penrose AF condition is too strong for our purposes: we cannot demand that the conformally rescaled field be well defined on $\mathscr{I}^+$. We can, however, weaken this condition in a natural way by demanding, instead, that the 'projections' of the field on $\mathscr{I}^+$ be well defined. By this we mean that if $\hat{X}^a_\mu(x)$ ($\mu = 0, 1, 2$) is a smooth triad field on the conformally rescaled space which lies entirely in the tangent space of $\mathscr{I}^+$ when $x \in \mathscr{I}^+$, then the projections

$$\hat{W}_{\alpha\ldots\omega} = \hat{W}_{ab\ldots z}\hat{X}^a_\alpha \ldots \hat{X}^z_\omega$$

are well defined on $\mathscr{I}^+$. Here $\hat{W}_{a\ldots z} = \hat{\phi}_{A\ldots Z}\hat{\varepsilon}_{A'B'}\ldots\hat{\varepsilon}_{Y'Z'}$ is the self-dual tensor field associated with $\hat{\phi}$. In particular, if we choose $\hat{X}^a_\mu$ such that

$$\hat{X}^a_0 = \hat{\iota}^A\hat{\bar{\iota}}^{A'} = \hat{n}^a, \qquad \hat{X}^a_1 = \hat{o}^A\hat{\bar{\iota}}^{A'} = \hat{m}^a, \qquad \hat{X}^a_2 = \hat{\iota}^A\hat{\bar{o}}^{A'} = \hat{\bar{m}}^a,$$

where $(\hat{o}_A, \hat{\iota}_A)$ is the conformally rescaled spinor dyad field associated with the rescaled coordinate system $(u, \Omega, \zeta, \bar{\zeta})$ $(\Omega = -1/r)$, we see that the limits

$$\lim_{\Omega\to 0}\hat{\phi}_i = \phi^0_i \tag{1.6}$$

exist for $2s \geq i \geq s$. The corresponding limits for the remaining components will in general not exist since they involve the vector $\hat{l}^a = \hat{o}^A\hat{\bar{o}}^{A'}$, which, unlike $\hat{n}^a$, $\hat{m}^a$ and $\hat{\bar{m}}^a$, does not lie in the tangent space of $\mathscr{I}^+$.

Condition (1.6) can be strengthened by demanding that $\phi^0_i(u, \zeta, \bar{\zeta})$ be smooth functions on $\mathscr{I}^+$ and that $\hat{\phi}_i$ be 'smoothly' attached to $\mathscr{I}^+$ in the sense that

$$\lim_{\Omega\to 0}(\Delta\hat{\phi}_i) = \Delta\phi^0_i), \tag{1.7}$$

where $\Delta$ is any combination of $u$, $\zeta$ and $\zeta$ derivatives. Assuming that (1.6) and (1.7) hold in any conformal frame, one can deduce that

$$\lim_{\Omega\to 0}\Omega^n\frac{\partial^n}{\partial\Omega^n}\hat{\phi}_i = 0 \tag{1.8}$$

for for $n \geq 1$ and $2s \geq i \geq s$.

In terms of the physical field components, our new AF conditions (1.6), (1.7) and (1.8) can be shown to be equivalent to

$$\begin{aligned} \phi_i &= O^*(r^{-(s+1+\epsilon)}) && \text{for } 0 \leq i < s, \\ \phi_i &= \phi_i r^{-2s-1+i} + O^*(r^{-2s-1-\epsilon+i}) && \text{for } s \leq i \leq 2s. \end{aligned} \tag{1.9}$$

In fact, by using the field equations, one can show that (1.9) are equivalent to the single condition

$$\phi_s = \phi^0_s r^{-(s+1)} + O^*(r^{-(s+1+\epsilon)}). \tag{1.10}$$

On comparing (1.9) with (1.5) and (1.4), we see that our new AF condition is sufficiently weak to include radiation fields of the type considered above. From now on, a field will be said to be AF if it satisfies condition (1.10).

As we shall see in § 3, a field which is AF at $\mathscr{I}^+$ need not be AF at $\mathscr{I}^-$, nor well behaved at space-like infinity. In order to avoid such fields, we need an extra AF condition which should, of course, be sufficiently weak to include the above examples of radiation fields. Such a condition is given by

$$\phi^0_s(u, \zeta, \zeta) = \phi(\zeta, \bar{\zeta}) + O^*(u^{-\epsilon}) \tag{1.11}$$

for large *negative u*. In the following sections we shall show that fields which satisfy both (1.10) and (1.11) are AF at both $\mathscr{I}^+$ *and* $\mathscr{I}^-$, well behaved at space-like infinity, and carry finite momentum and angular momentum.

Even with this extra condition our field is not necessarily as well behaved as we might wish. For example, given a Maxwell field which satisfies (1.11) and (1.10), it is comparatively simple, using the energy–momentum tensor $T_{ab} = \phi_{AB}\phi_{A'B'}$, to calculate the total flux of momentum $P_a^{+(-)}$ and angular momentum $M_{ab}^{+(-)}$ radiated through $\mathscr{I}^{+(-)}$. These quantities are finite and, as one might expect,

$$P_a^+ = P_a^-.$$

However, if $\phi$ given by (1.11) has an imaginary component then

$$M_{ab}^+ \neq M_{ab}^-.$$

The third AF condition, $\phi$ = real, therefore appears to be necessary. This is satisfied by our examples of radiation fields and, as we shall see in the following sections, it guarantees that $M_{ab}^+ = M_{ab}^-$ for fields of arbitrary (integer) spin.

Thus, in order to obtain a class of fields which is sufficiently general to include radiation fields of the type considered and yet, at the same time, sufficiently restricted for quantities such as total momentum and angular momentum to be finite and uniquely defined, the following AF conditions are needed:

$$\begin{aligned} \phi_s &= \phi_s^0 r^{-(s+1)} + \mathrm{O}^*(r^{-(s+1+\epsilon)}), \\ \phi_s^0 &= \phi + \mathrm{O}^*(u^{-\epsilon}) \qquad \text{for negative } u, \end{aligned} \tag{1.12}$$

where $\phi(\zeta, \bar{\zeta})$ is real. We shall refer to fields which satisfy (1.12) as radiation fields, and in the remainder of this paper we shall consider their properties.

After developing a certain amount of mathematical machinery in § 2 we shall, in § 3, show that radiation fields are AF at $\mathscr{I}^-$ as well as at $\mathscr{I}^+$, and that they can be put into one-to-one correspondence with a certain class of weighted functions defined on either $\mathscr{I}^+$ or $\mathscr{I}^-$. In § 4 we show that there exists a natural symplectic product between any two spin-$s$ radiation fields which may be written as an integral involving these weighted functions over $\mathscr{I}^+$ or $\mathscr{I}^-$. We then use this symplectic product to define the total momentum and angular momentum carried by a radiation field. Finally, in § 5, we discuss the relevance of these fields to certain curved space and nonlinear systems.

## 2. Radially null coordinate systems

In this section, which is included mainly for the sake of completeness and to fix a consistent notation, we shall review some standard work on radially null coordinate systems, spin and conformally weighted functions and the differential operators $\eth$ and $\bar{\eth}$.

We begin by introducing a standard prescription by means of which a unique, retarded (or advanced), radially null coordinate system $(u, r, \zeta, \bar{\zeta})$ can be constructed from a given reference frame consisting of a constant spinor dyad field $\{\alpha_A, \beta_A\}$ $(\alpha_A\beta^A = 1)$ together with an origin point 0. This prescription is useful because it provides us with an isomorphism between the Poincaré transformations connecting such frames and the corresponding coordinate transformations connecting the associated coordinate systems.

We first define a family of spinors according to

$$O_A(\zeta,\zeta)=(2P)^{-1/2}(\alpha_A+\bar{\zeta}\beta_A) \tag{2.1}$$

where

$$P=\tfrac{1}{2}(1+\zeta\bar{\zeta}) \tag{2.2}$$

and $\zeta$ is a complex number which we interpret as a stereographic coordinate labelling points on a sphere. This family of spinors defines a family of null vectors $L_a(\zeta,\bar{\zeta})=O_A\bar{O}_{A'}$, which sweep out all null directions as $\zeta$ varies over the sphere, and which automatically satisfy

$$L_a v^a=1, \tag{2.3}$$

where

$$v^a=\alpha^A\alpha^{A'}+\beta^A\beta^{A'}. \tag{2.4}$$

Since $\alpha^A\beta_A=1$, $v^a$ satisfies

$$v^a v_a=2. \tag{2.5}$$

For future reference we also define a related family of spinors by

$$I_A=-v_A^{A'}\bar{O}_{A'}. \tag{2.6}$$

From (2.3) it follows immediately that

$$O_A I^A=1. \tag{2.7}$$

Given any other dyad $\{\alpha^*_A,\beta^*_A\}$, related to $\{\alpha_A,\beta_A\}$ by an SL(2, $C$) transformation

$$\begin{aligned}\alpha_A&=a\alpha^*_A+b\beta^*_A,\\ \beta_A&=c\alpha^*_A+d\beta^*_A,\end{aligned}\qquad ad-cb=1, \tag{2.8}$$

we can construct the associated quantities

$$O^*_A=(2P^*)^{-1/2}(\alpha^*_A+\bar{\zeta}^*\beta^*_A) \tag{2.9}$$

and

$$v^*_{AA'}=\alpha^*_A\alpha^*_{A'}+\beta^*_A\beta^*_{A'}.$$

From these relations we see that, when $O_A$ and $O^*_A$ are parallel, $\zeta$ and $\zeta^*$ are related by the bilinear transformation

$$\bar{\zeta}^*=(b+d\bar{\zeta})/(a+c\bar{\zeta}) \tag{2.10}$$

and that

$$O^*_A=K^{1/2}\,\mathrm{e}^{\mathrm{i}\lambda/2}\,O_A, \tag{2.11}$$

where $K$ and $\lambda$ are given by

$$K=v^*_a L^a \tag{2.12}$$

and

$$\partial\zeta/\partial\zeta^*=\mathrm{e}^{\mathrm{i}\lambda}K^{-1}P/P^*. \tag{2.13}$$

In particular, under a U(2) rotation we have

$$v^{a^*}=v^a,\qquad K=1, \tag{2.14}$$

and

$$\partial\zeta/\partial\zeta^* = e^{i\lambda} P/P^*. \tag{2.15}$$

These relations will prove useful when we come to consider spin and conformally weighted functions.

Since $L^a$ sweeps out all null directions, the position vector $x^a$ of some point $P$ relative to an origin point 0 can be written as

$$x^a = rL^a(\zeta, \bar{\zeta}) + uv^a \qquad (r \geqslant 0) \tag{2.16}$$

for some unique values of $u$, $r$ and $\zeta$. Thus $(u, r, \zeta, \bar{\zeta})$ serves as a (retarded) radially null coordinate system and is uniquely determined by the frame $\{\alpha_A, \beta_A, 0\}$. The $u$ = constant surfaces are future null cones emanating from the line $x^a = uv^a$, $\zeta$ labels the generators of these cones, and $r$ is an affine parameter along the generators. (In a similar way we can define an advanced system $(u', r', \zeta', \zeta')$ according to

$$x^a = r'L^{a'}(\zeta', \bar{\zeta}') + u'v^{a'} \tag{2.17}$$

where $L^{a'} = -L^a$ and $v^{a'} = v^a$.)

The spinor dyad field associated with such a coordinate system is given by

$$o_A(u, r, \zeta, \bar{\zeta}) = O_A(\zeta, \bar{\zeta}), \qquad \iota_A(u, r, \zeta, \bar{\zeta}) = I_A(\zeta, \bar{\zeta}), \tag{2.18}$$

and, in accordance with the standard spin coefficient notation, the independent components of the field $\phi_{A\ldots Z}$ are defined by

$$\phi_i(u, r, \zeta, \bar{\zeta}) = \phi_{A\ldots Z} \underbrace{\iota^A \ldots \iota^L}_{i} \underbrace{o^M \ldots o^Z}_{2s-i}. \tag{2.19}$$

Given any other reference frame $\{\alpha^*_A, \beta^*_A, 0^*\}$ we can, by the same prescription, construct a new coordinate system $(u^*, r^*, \zeta^*, \bar{\zeta}^*)$ together with its associated spinor dyad field $\{o^*_A, \iota^*_A\}$, and find the corresponding components $\phi^*_i$ of the field. If our two frames are related by a U(2) rotation we have

$$\phi^*_k(u^*, \zeta^*, \bar{\zeta}^*) = e^{i(s-k)\lambda}\phi_k(u, \zeta, \bar{\zeta}) \tag{2.20}$$

where $\lambda$ is determined by equation (2.13). In general, a quantity $\eta$ which transforms according to $\eta^* = e^{is\lambda}\eta$ under a U(2) rotation is said to have spin weight (sw) $s$. Thus $\phi_i$ has sw $(s-i)$. Two differential operators associated with spin weighted functions are given by

$$\eth\eta = 2P^{1-s}\,\partial(P^s\eta)/\partial\zeta, \qquad \bar{\eth}\eta = 2P^{1+s}\,\partial(P^{-s}\eta)/\partial\bar{\zeta}, \tag{2.21}$$

where $\eta$ has sw $s$ (Newman and Penrose 1966). Using (2.15) one can show that $\eth\eta$ has sw $s+1$ and $\bar{\eth}\eta$ has sw $s-1$.

In terms of the above notation, the field equations can be shown to be equivalent to

$$\dot{\phi}_i - \frac{\partial\phi_i}{\partial r} - \frac{i+1}{r}\phi_i = -\frac{1}{r}\eth\,\phi_{i+1}, \tag{2.22}$$

$$\frac{\partial\phi_{i+1}}{\partial r} + \frac{2s-i}{r}\phi_{i+1} = -\frac{1}{r}\bar{\eth}\,\phi_i, \tag{2.23}$$

where $\cdot \equiv \partial/\partial u$ (Couch and Newman 1972). Using the radial field equations (2.23), together with the AF condition (1.10), the asymptotic fall-off behaviour of the field components $\phi_i$ can be determined and is given by

$$\phi_i = \phi_i^0(u, \zeta, \bar{\zeta})r^{-2s-1+i} + O^*(r^{-2s-1+i-\epsilon}) \qquad \text{for } 2s \geq i \geq s, \tag{2.24}$$

and

$$\phi_i = O^*(r^{-(s+1+\epsilon)}) \qquad \text{for } s > i \geq 0. \tag{2.25}$$

The remaining set of field equations (2.22) can now be used to show that the asymptotic field components $\phi_i^0$ satisfy

$$\dot{\phi}_i^0 = -\eth\phi_{i+1}^0. \tag{2.26}$$

Equations (2.26) are sometimes referred to as the asymptotic field equations on $\mathscr{I}^+$. (The corresponding equations on $\mathscr{I}^-$, which can be obtained by using an advanced coordinate system, take the same form.)

If the coordinate systems $(u, r, \zeta, \bar{\zeta})$ and $(u^*, r^*, \zeta^*, \bar{\zeta}^*)$ are related by a pure translation of origin point given by $\gamma^a$, then the asymptotic field component $\phi_{2s}^0$ transforms according to

$$\phi_{2s}^{0*}(u^*, \zeta^*, \bar{\zeta}^*) = \phi_{2s}^0(u, \zeta, \bar{\zeta}) \tag{2.27}$$

where

$$u^* = u + L_a\gamma^a \qquad \text{and} \qquad \zeta^* = \zeta. \tag{2.28}$$

Similarly, under a pure SL(2, $C$) transformation, we have

$$\phi_{2s}^{0*}(u^*, \zeta^*, \bar{\zeta}^*) = K^{-(s+1)}\,\mathrm{e}^{-is\lambda}\,\phi_{2s}^0(u, \zeta, \bar{\zeta}) \tag{2.29}$$

where

$$u^* = Ku \tag{2.30}$$

and $\zeta^*$ and $\zeta$ are related by equation (2.10).

The transformation laws for the remaining asymptotic field components assume a more complicated form and will not be considered here.

The function $\phi_{2s}^0$ is an example of a quantity with conformal weight as well as spin weight. In general we say that a function $\eta$ which transforms according to

$$\eta^* = K^w\,\mathrm{e}^{is\eta}\,\eta \tag{2.31}$$

under an SL(2, $C$) transformation has SW $s$ and conformal weight (CW) $w$, or weight $(s, w)$, for short. From (2.11) we see that $O_A$ has weight $(\frac{1}{2}\frac{1}{2})$ and, on using (2.13), that the spherical surface element

$$\mathrm{d}\Omega = \mathrm{d}\zeta\Lambda\mathrm{d}\bar{\zeta}/8\pi\mathrm{i}P^2 \tag{2.32}$$

has weight $(0, 2)$. From (2.30) we also see that a $u$ derivative decreases conformal weight by unity: if $\eta$ has weight $(s, w)$ then $\dot{\eta}$ has weight $(s, w-1)$.

Finally, to conclude this section, we consider the transformation properties of weighted functions under infinitesimal Poincaré transformations. Under an infinitesimal translation, $e^a$, we have

$$\mathscr{L}\eta = (\mathscr{L}_a e^a)\eta = \eta^*(u, \zeta, \bar{\zeta}) - \eta(u, \zeta, \bar{\zeta}) = \dot{\eta}L_a e^a \tag{2.33}$$

where $\mathscr{L}_a$ is the translation Lie derivative. Similarly, under an infinitesimal Lorentz transformation determined by $e^{[ab]}$ we can define the 'Lorentz' Lie derivative $\mathscr{L}_{ab}$ by

$$(\mathscr{L}_{ab}e^{ab}\eta) = \eta^*(u, \zeta, \bar{\zeta}) - \eta(u\,\zeta\,\bar{\zeta}).$$

However, for our purpose, it is more convenient to use complex self-dual Lorentz transformations of the form $SL(2, C) \otimes 1$, rather than real transformations which have the form $SL(2, C) \otimes \overline{SL(2, C)}$. These complex transformations do not affect dashed spinors and, in infinitesimal form, are determined by a symmetric spinor $e^{AB}$. The corresponding real transformation is determined by $e^{ab} = \varepsilon^{A'B'}e^{AB} + \varepsilon^{AB}e^{A'B'}$. Using (2.30) and (2.31), one can show that the Lie derivative associated with these transformations is given by

$$\mathscr{L}_{AB}\eta = \bar{\eth}\eta O_A O_B - [u\dot{\eta} - (w+s)\eta]O_{(A}I_{B)}. \tag{2.34}$$

It is easily checked that $\mathscr{L}_{AB}$ preserves spin and conformal weight. Relations (2.33) and (2.34) will be used in § 4 to define the total momentum and angular momentum carried by a spin-$s$ radiation field.

## 3. The asymptotic field components

A characteristic property of free, AF, massless fields, which we shall prove in the Appendix, is that they satisfy the following form of the Kirchhoff–Penrose integral theorem:

$$\phi_{A\ldots Z}(x^a) = -\oint \dot{\phi}^0_{2s}(x^aL_a, \zeta, \bar{\zeta})O_A \ldots O_Z \,\mathrm{d}\Omega \tag{3.1}$$

where $\phi^0_{2s}$ is the asymptotic $r^{-1}$ component of the field, and $\mathrm{d}\Omega$ is the spherical surface element (2.32). Note that (3.1) is form invariant because the sum of the weights of all its components is zero. According to (3.1), an AF, spin-$s$ field is completely determined by a function, namely $\dot{\phi}_{2s}$, of weight $(-s, -(s+2))$. Conversely, according to the results of the previous section, a free AF, spin-$s$ field determines a function of weight $(-s, -(s+2))$, namely its asymptotic component $\dot{\phi}^0_{2s}$. We therefore see that a two-way correspondence exists between AF, spin-$s$ fields and functions of weight $(-s, -(s+2))$. Such a function cannot, however, be specified arbitrarily; the requirement that the field be AF imposes a restriction on its asymptotic behaviour for large $u$. In this section we shall determine this asymptotic behaviour and the corresponding behaviour of the asymptotic components $\phi^0_i$ of the field on both $\mathscr{I}^+$ and $\mathscr{I}^-$.

We begin by proving the following theorem which gives a sufficient condition for the field associated with a function of weight $(-s, -(s+2))$ to be AF in the sense of (1.10).

*Theorem 1.* If $\chi(u, \zeta, \bar{\zeta})$ has weight $(-s, -(s+2))$ and satisfies $\chi = O^*(u^{-(s+1+\epsilon)})$ for large *positive* $u$, then the field $\psi_{A\ldots Z}(x^a)$, defined by

$$\psi_{A\ldots Z} = -\oint \chi(x^aL_a, \zeta, \bar{\zeta})O_A \ldots O_Z \,\mathrm{d}\Omega, \tag{3.2}$$

is AF at $\mathscr{I}^+$ and its asymptotic components $\psi^0_i(u, \zeta, \bar{\zeta})$ on $\mathscr{I}^+$ are given by

$$\psi^0_i = (-1)^i(\partial^{i-s}/\partial u^{i-s})\,\bar{\eth}^{2s-i}S \qquad (2s \geq i \geq s) \tag{3.3}$$

where $S(u, \zeta, \bar{\zeta})$ is the unique function of weight $(-s, -1)$ which satisfies

$$\partial^{s+1}S/\partial u^{s+1} = \chi \tag{3.4}$$

and

$$\lim_{u\to\infty} S = 0. \tag{3.5}$$

Similarly, if $\chi = O^*(u^{-(s+1+\epsilon)})$ for large *negative* $u$, then the field given by (3.2) is AF at $\mathscr{I}^-$ and its asymptotic components $\psi_i^{0'}(u, \zeta, \bar{\zeta})$ on $\mathscr{I}^-$ are given by

$$\psi_i^{0'} = (-1)^i(\partial^{i-s}/\partial u^{i-s})\bar{\eth}^{2s-i}S' \tag{3.6}$$

where

$$\partial^{s+1}S'/\partial u^{s+1} = \chi \tag{3.7}$$

and

$$\lim_{u\to-\infty} S' = 0. \tag{3.8}$$

*Proof.* First of all, it is clear that $\psi$ is automatically a solution of the field equations, for

$$\nabla^A_{A'}\psi_{A\ldots Z} = -\oint \dot{\chi}O^A O_{A'} O_A \ldots O_Z \, d\Omega = 0.$$

In order to prove asymptotic flatness at $\mathscr{I}^+$ it is sufficient to show that

$$r^{s+1}\psi_s = \psi_s^0 + O^*(r^{-\epsilon}) \tag{3.9}$$

for any given outgoing null geodesic. Moreover, since (3.2) is form invariant, we may, without loss of generality, choose our coordinate system such that this given geodesic is specified by $\zeta = u = 0$. In this case (3.2) gives

$$r^{s+1}\psi_s(0, r, 0, 0) = r^{s+1}\psi_s(r) = -r^{s+1}\oint \chi\left(\frac{r\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}, \zeta, \bar{\zeta}\right)\frac{\bar{\zeta}^a}{P^s}\, d\Omega \tag{3.10}$$

where we have used equations (2.1), (2.6) and (2.16). We require to show that $r^{s+1}\psi_s$, given by (3.10), satisfies (3.9).

Consider the regular sw $-s$ function $X$ defined by

$$\chi(u, \zeta, \bar{\zeta}) - \frac{1}{P^s}\sum_0^s \frac{\eth^i\chi(u, 0, 0)\zeta^i}{i!} = X(u, \zeta, \bar{\zeta}).$$

By making a Taylor series expansion about $\zeta = \bar{\zeta} = 0$, it can be seen that $X$ has the form

$$X = (\zeta^{s+1}\bar{\zeta}/P^{s+1})Y(u, \zeta, \bar{\zeta}),$$

where $Y$ is a regular (sw 0) function. We can therefore write $\chi$ in the form

$$\chi = \frac{1}{P^s}\sum_0^s \frac{\eth^i\chi(u, 0\, 0)\zeta^i}{i!} + \frac{\zeta^{s+1}\bar{\zeta}}{P^{s+1}}Y(u, \zeta, \bar{\zeta}). \tag{3.11}$$

Since $\chi = O^*(u^{-(s+1+\epsilon)})$, it is clear that

$$\eth^i\chi(u, 0\, 0) = O^*(u^{-(s+1+\epsilon)}) \qquad \text{and} \qquad Y(u, \zeta, \bar{\zeta}) = O^*(u^{-(s+1+\epsilon)}). \tag{3.12}$$

By substituting (3.11) into (3.10) and making a change of variables from $\zeta$ and $\bar{\zeta}$ to $u$ and $\phi$, where

$$u = r\zeta\bar{\zeta}/(1+\zeta\bar{\zeta}) \qquad \text{and} \qquad e^{2i\phi} = \zeta/\bar{\zeta},$$

we obtain

$$r^{s+1}\psi_s = -\int_0^r \left(\frac{\eth^s\chi(u, 0, 0)}{s!} + \frac{u}{r}Y_0\right)u^s\left(1-\frac{u}{r}\right)^s du, \tag{3.13}$$

where

$$Y_0(u)=\int_0^{2\pi} Y\,\mathrm{d}\phi.$$

Equation (3.13) together with the asymptotic conditions (3.10) imply that

$$r^{s+1}\psi_s=\psi_s^0+\mathrm{O}^*(r^{-\epsilon}),$$

where $\psi_s^0$ is finite and given by

$$\psi_s^0=-\int_0^{\infty}\frac{\eth^s\chi(u,0,0)u^s}{s!}\,\mathrm{d}u. \tag{3.14}$$

The field is thus AF at $\mathscr{I}^+$.

Using equation (3.14) together with the asymptotic field equations (2.27), it is now an easy matter to deduce that the asymptotic components, $\psi_i^0$ for $2s\geqslant i\geqslant s$, of the field are given by

$$\psi_i^0=(-1)^i(\partial^{i-s}/\partial u^{i-s})\eth^{2s-i}S$$

where $S$ is determined by

$$\partial^{s+i}S/\partial u^{s+1}=\chi \qquad \text{and} \qquad \lim_{u\to\infty}S=0.$$

The proof of the remaining part of the theorem relating to $\mathscr{I}^-$ is the same as that given above except that an advanced coordinate system is used in place of a retarded one.

According to this theorem, asymptotic flatness at $\mathscr{I}^+$ depends only on the behaviour of $\chi$ for large *positive* $u$; the behaviour of $\chi$ for large negative $u$ does not affect asymptotic flatness at $\mathscr{I}^+$.

We have shown that the asymptotic components of a field generated by a function $\chi$ which satisfies $\chi=\mathrm{O}^*(u^{-(s+1+\epsilon)})$ are given by (3.3) where $S$ satisfies $S=\mathrm{O}^*(u^{-\epsilon})$. Our next theorem shows that this is true for *all* AF fields.

*Theorem 2.* If a spin-$s$ field is AF at $\mathscr{I}^+$, then its asymptotic components $\phi_i^0$ on $\mathscr{I}^+$ have the form

$$\phi_i^0=(-1)^i(\partial^{i-s}/\partial u^{i-s})\,\eth^{2s-i}S$$

where $S$ is a function of weight $(-s,-1)$ which satisfies $S=\mathrm{O}^*(u^{-\epsilon})$ for large *positive* $u$. Similarly, if the field is AF at $\mathscr{I}^-$, then is asymptotic components $\phi_i^{0'}$ on $\mathscr{I}^-$ have the form

$$\phi_i^{0'}=(-1)^i(\partial^{i-s}/\partial u^{i-s})\,\eth^{2s-i}S'$$

where $S'$ satisfies $S'=\mathrm{O}^*(u^{-\epsilon})$ for large *negative* $u$.

*Proof.* Let $\theta(u)$ be a smooth function with the property that $\theta(u)=1$ for $u\leqslant R$, and $\theta(u)=0$ for $u>R+1$, and let $\chi=\theta\dot{\phi}_{2s}$. $\chi$ obviously satisfies the requirements of theorem 1, and from the KP integral theorem it is clear that

$$\psi_s=\phi_s \tag{3.15}$$

on the origin cone, provided that $r\leqslant R$. Since $\phi$ and $\psi$ are both AF we have, on the origin cone,

$$R^{s+1}\psi_s(R)=\psi_s^0+\mathrm{O}^*(R^{-\epsilon}),\qquad R^{s+1}\phi_s(R)=R^{s+1}\psi_s(R)=\phi_s^0+\mathrm{O}^*(R^{-\epsilon}),$$

and hence

$$\phi_s^0=\psi_s^0+\mathrm{O}^*(R^{-\epsilon}).$$

Thus, on using (3.14), we have

$$\phi_s^0=-\int_0^\infty \frac{\theta\eth^s\dot{\phi}_{2s}u^s}{s!}\,\mathrm{d}u+\mathrm{O}^*(R^{-\epsilon})$$

and hence

$$\phi_s^0=-\lim_{R\to\infty}\frac{\int_0^\infty \theta\eth^s\dot{\phi}_{2s}^0u^s}{s!}\,\mathrm{d}u=\frac{-\int_0^\infty \eth^s\dot{\phi}_{2s}^0u^s}{s!}\,\mathrm{d}u. \tag{3.16}$$

Since $\phi_s^0$ is finite and $\phi_{2s}^0$ is a smooth function of sw $-s$ it can be seen that equation (3.16) implies that $\dot{\phi}_{2s}^0$ and all its $\eth$ derivatives must fall like $\mathrm{O}(u^{-(s+1+\epsilon)})$ for large positive $u$. Furthermore, by making use of the fact that this applies for all coordinate frames, one can show that the $\eth$ derivatives of $\dot{\phi}_{2s}^0$ transform correctly if and only if $\dot{\phi}_{2s}^0=\mathrm{O}^*(u^{-(s+1+\epsilon)})$. The remainder of the proof now follows directly the KP integral theorem and theorem 1.

Let us now apply our theorems to a spin-$s$ radiation field which, by definition, satisfies

$$\phi_s=\phi_s^0r^{-(s+1)}+\mathrm{O}^*(r^{-(s+1+\epsilon)})\qquad\text{and}\qquad \phi_s^0=\phi(\zeta,\bar{\zeta})+\mathrm{O}^*(u^{-\epsilon}) \tag{3.17}$$

for large negative $u$, where $\phi$ is real. These conditions together with our theorems can be seen to imply:

(1) The field is AF at *both* $\mathscr{I}^+$ and $\mathscr{I}^-$.

(2) From the point of view of $\mathscr{I}^+$, the field is uniquely associated with a function $S$ of weight $(-s,-1)$ which satisfies:

(a) $S=\mathrm{O}^*(u^{-\epsilon})$ for large positive $u$; (3.18)

(b) $S=\pi(\zeta,\bar{\zeta})+\mathrm{O}^*(u^{-\epsilon})$, for large negative $u$, (3.19)

where $\pi$ is given by

$$(-1)^s\eth^s\pi=\phi; \tag{3.20}$$

(c) $\phi_i^0=(-1)^i(\partial^{i-s}/\partial u^{i-s})\,\eth^{2s-i}S.$ (3.21)

(3) From the point of view of $\mathscr{I}^-$, the field is uniquely associated with a function $S'$ of weight $(-s,-1)$ which satisfies:

(a) $S'=\mathrm{O}^*(u^{-\epsilon})$ for large negative $u$; (3.22)

(b) $S'=-\pi(\zeta,\bar{\zeta})+\mathrm{O}^*(u^{-\epsilon})$ for large positive $u$; (3.23)

(c) $\phi_i^{0'}=(-1)^i(\partial^{i-s}/\partial u^{i-s})\eth^{2s-i}S'.$ (3.24)

(4) $S'(u,\zeta,\bar{\zeta})=S(u,\zeta,\bar{\zeta})-\pi.$ (3.25)

Furthermore, by means of the KP integral theorem one can show that the field is AF and purely electric at space-like infinity in the sense specified by Ashtekar and Hansen (1978).

Finally, we remark that since $\pi$ is a regular sw $-s$ function for which $\eth^s\pi$ is real (cf equation (3.20)), the theory of spin weighted functions (Newman and Penrose 1966) can be used to show that

$$\pi = \bar{\eth}^s X \tag{3.26}$$

for some real sw 0 function $X$. Furthermore, $X$ may be chosen such that it consists only of spherical harmonic functions $Y_{lm}$ for which $l \geqslant s$, or, equivalently, that $P^s X$ is a regular function where $P$ is given by equation (2.2). Equations (3.19) and (3.25) thus become

$$S = \bar{\eth}^s X + O^*(u^{-\epsilon}) \qquad \text{(for negative } u\text{)}$$

and (3.27)

$$S' = S = \bar{\eth}^s X$$

where $P^5 X$ is regular.

Equations (3.27) will prove useful when we come to consider angular momentum.

## 4. Momentum and angular momentum

In this section we shall first determine expressions for the momentum $P_a^{+(-)}$ and angular momentum $M_{ab}^{+(-)}$ radiated through $\mathscr{I}^{+(-)}$ by a spin-$s$ radiation field, and then show that they satisfy

$$P_a^+ = P_a^- = P_a = \tfrac{1}{2}\Omega(S, \mathscr{L}_a S)$$

and

$$M_{ab}^+ = M_{ab}^- = M_{ab} = \tfrac{1}{2}\Omega(S, \mathscr{L}_{ab} S)$$

where $\Omega(S_1, S_2)$ is a well defined symplectic product between spin-$s$ radiation fields.

We start with spin-1 (Maxwell) radiation fields which, unlike fields of higher spin, possess a well defined energy–momentum tensor given by

$$T_{ab} = \phi_{AB}\phi_{A'B'}. \tag{4.1}$$

In terms of this tensor, the momentum and angular momentum radiated through $\mathscr{I}^+$ is given by

$$P_a^+ = \lim_{R\to\infty} \int_V T_{ca}\, \mathrm{d}V^c \tag{4.2}$$

and

$$M_{ab}^+ = \lim_{R\to\infty} \int_V T_{c[a}\, x_{b]}\, \mathrm{d}V^c$$

where $V$ is the hypersurface $r = R$. In terms of our notation it is, however, more convenient to use the symmetric spinor component $\overset{+}{\omega}_{AB}$ of $M_{ab}^+$, where

$$M_{ab}^+ = \varepsilon_{A'B'}\overset{+}{\omega}_{AB} + \varepsilon_{AB}\overset{+}{\omega}_{A'B'}.$$

In terms of $T_{ab}$, $\overset{+}{\omega}_{AB}$ is given by

$$\overset{+}{\omega}_{AB} = \lim_{R\to\infty} \int_V T_{CC'A'(A}\, x_{B)}^{A'}\, \mathrm{d}V^{CC'}. \tag{4.3}$$

By means of equations (4.1). (4.2), (4.3), (2.16) and (2.24) one can easily show that

$$P_a^+ = \int \bar{\phi}_2^0 \phi_2^0 L_a \, d\Omega \, du \tag{4.4}$$

and

$$\omega_{AB}^+ = -\int \{\bar{\phi}_1^0 O_A O_B + u\bar{\phi}_2^0 O_{(A|B)}\}\phi_2^0 \, d\Omega \, du. \tag{4.5}$$

For spin-1 fields we have $\phi_2^0 = \dot{S}$ and $\phi_1^0 = -\eth S$ where $S$ has weight $(-1, -1)$. Thus, when expressed in terms of $S$, equations (4.4) and (4.5) give

$$P_a^+ = \int \dot{\bar{S}}\dot{S} L_a \, d\Omega \, du = \int (\mathscr{L}_a \bar{S})\dot{S} \, d\Omega \, du \tag{4.6}$$

and

$$\omega_{AB}^+ = \int \{\bar{\eth}\bar{S} O_A O_B - u\dot{\bar{S}} O_{(A|B)}\}\dot{S} \, d\Omega \, du = \int (\mathscr{L}_{AB}\bar{S})\dot{S} \, d\Omega \, du, \tag{4.7}$$

where we have used the Lie derivatives defined by equations (2.33) and (2.34). Similarly for $\mathscr{I}^-$, we have

$$P_a^- = \int (\mathscr{L}_a \bar{S}')\dot{S}' \, d\Omega \, du, \tag{4.8}$$

$$\omega_{AB}^- = \int (\mathscr{L}_{AB}\bar{S}')\dot{S}' \, d\Omega \, du. \tag{4.9}$$

In order to show that

$$P_a^+ = P_a^- \tag{4.10}$$

and

$$\omega_{AB}^+ = \omega_{AB}^- \tag{4.11}$$

we use the fact that

$$S' = S - \bar{\eth}X, \tag{4.12}$$

where $X$ is real (cf equation (3.27)). Since $\dot{S} = \dot{S}'$, the proof of (4.10) is obvious. The proof of (4.11) is more involved and relies on the reality of $X$. From equations (4.7), (4.9) and (4.12), we have

$$\omega_{AB}^+ - \omega_{AB}^- = \oint \bar{\eth}X \eth\bar{\eth}X O_A O_B \, d\Omega$$

$$= \tfrac{1}{2}\oint \eth(\bar{\eth}X\bar{\eth}X) O_A O_B \, d\Omega = \tfrac{1}{2}\oint \bar{\eth}X\bar{\eth}X \eth(O_A O_B) \, d\Omega = 0,$$

where we have integrated by parts and used the fact that $\eth O_A = 0$.

Since fields of higher spin do not possess a well defined energy–momentum tensor, the above procedure cannot be used to determine expressions for their momentum and angular momentum. Nevertheless, by analogy with the spin-1 case, we can simply define the $\omega_{AB}^{+(-)}$ and $P_a^{+(-)}$ of a spin-$s$ field by

$$P_a^+ = \int (\mathscr{L}_a \bar{S})\dot{S} \, d\Omega \, du, \qquad P_a^- = \int (\mathscr{L}_a \bar{S}')\dot{S}' \, d\Omega \, du, \tag{4.13}$$

$$\omega_{AB}^+ = \int (\mathscr{L}_{AB}\bar{S})\dot{S} \, d\Omega \, du, \qquad \omega_{AB}^- = \int (\mathscr{L}_{AB}\bar{S}')\dot{S}' \, d\Omega \, du, \tag{4.14}$$

where $S$ is now the associated function of weight $(-s, -1)$. These expressions are invariant under Lorentz transformations and transform according to

$$P_a^* = P_a, \tag{4.15}$$

$$\omega_{AB}^* = \omega_{AB} + P_{A'(A}x_{B)}^{A'} \tag{4.16}$$

under a pure translation of origin point, $00^* = x^a$.

In terms of vector notation, equation (4.16) gives

$$M_{ab}^* = M_{ab} + 2P_{[a}x_{b]}$$

which is, of course, the 'correct' transformation law for angular momentum.

We shall now show that the expressions (4.13) and (4.14) automatically satisfy (4.10) and (4.11). We use equation (3.27), namely

$$S' = S - \bar{\eth}^s X, \tag{4.17}$$

where $X$ is real and $P^s X$ is regular. Since $\dot{S}' = \dot{S}$, the proof of (4.10) is again obvious. The proof of (4.11) is very much more involved and again relies on the reality of $X$. After integrating by parts and using equation (4.17), equations (4.14) give

$$\overset{+}{\omega}_{AB} - \overset{-}{\omega}_{AB} = \tfrac{1}{2}\oint [(1+s)\bar{\eth}\eth^S X \bar{\eth}^s X - (1-s)\eth^s X \bar{\eth}^{s+1} X] O_A O_B \, \mathrm{d}\Omega. \tag{4.18}$$

If we now substitute $X' = P^{s-1}X$ into this expression we obtain

$$\begin{aligned} \overset{+}{\omega}_{AB} - \overset{-}{\omega}_{AB} &= \tfrac{1}{2}\oint P^3 A O_A O_B \, \mathrm{d}\Omega \\ &= \tfrac{1}{2}\oint A(\alpha_A\alpha_B + 2\bar{\zeta}\alpha_{(A}\beta_{B)} + \bar{\zeta}^2\beta_A\beta_B)\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \end{aligned} \tag{4.19}$$

where

$$A = (1+s)\frac{\partial^{s+1}X'}{\partial\bar{\zeta}\partial\zeta^s}\frac{\partial^s X'}{\partial\bar{\zeta}^s} - (1-s)\frac{\partial^s X'}{\partial\zeta^s}\frac{\partial^{s+1}X'}{\partial\bar{\zeta}^{s+1}}.$$

After some algebraic manipulation, one can show that $A$ can be written in the form

$$A = \partial^3 B/\partial\bar{\zeta}^3 + \partial C/\partial\zeta \tag{4.20}$$

where $B$ and $C$ consist of linear combinations of terms like

$$\frac{\partial^{i+j}X'}{\partial\bar{\zeta}^i\partial\zeta^j}\cdot\frac{\partial^{k+l}X'}{\partial\bar{\zeta}^l\partial\zeta^k}.$$

Therefore, on substituting (4.20) into (4.19), integrating by parts and using the fact that $P^s X$ is regular, we obtain the required result, namely

$$\overset{+}{\omega}_{AB} - \overset{-}{\omega}_{AB} = 0. \tag{4.21}$$

From the point of view of Hamiltonian theory, a desirable property of $P_a$ and $\omega_{AB}$ is that they can be written in the form

$$P_a = \tfrac{1}{2}\Omega(S, \mathscr{L}_a S) \tag{4.22}$$

and

$$\omega_{AB} = \tfrac{1}{2}\Omega(S, \mathscr{L}_{AB} S), \tag{4.23}$$

where $\Omega(S_1, S_2)$ is the invariant, non-degenerate symplectic product defined by

$$\Omega(S_1, S_2) = \tfrac{1}{2} \int (S_2 \dot{\bar{S}}_1 - S_1 \dot{\bar{S}}_2 + \bar{S}_2 \dot{S}_1 - \bar{S}_1 \dot{S}_2)\, d\Omega\, du.$$

The proof of (4.22) is obvious; the proof of (4.23) goes as follows. By using equation (3.27) and integrating by parts, we have

$$\begin{aligned}\tfrac{1}{2}\Omega(S, \mathscr{L}_{AB}S) &= \tfrac{1}{4} \int [(\mathscr{L}_{AB}S)\dot{\bar{S}} - S\mathscr{L}_{AB}\dot{\bar{S}} + (\mathscr{L}_{AB}\bar{S})\dot{S} - \bar{S}\mathscr{L}_{AB}\dot{S}]\, d\Omega\, du \\ &= \tfrac{1}{2} \int [(\mathscr{L}_{AB}S)\dot{\bar{S}} + (\mathscr{L}_{AB}\bar{S})\dot{S}]\, d\Omega\, du \\ &= \int (\mathscr{L}_{AB}\bar{S})\dot{S}\, d\Omega\, du + \tfrac{1}{2} \oint [(1+s)\bar{\eth}\eth^s X \bar{\eth}^s X - (1-s)\eth^s X \bar{\eth}^{s+1} X] O_A O_B\, d\Omega.\end{aligned} \tag{4.24}$$

The first term on the right-hand side of (4.24) is $\omega_{AB}$, and the second term vanishes by equations (4.21) and (4.18). Thus

$$\omega_{AB} = \tfrac{1}{2}\Omega(S, \mathscr{L}_{AB}S).$$

## 5. Discussion

Even though this paper is firmly based in flat space-time and deals only with linear systems, many of the results obtained are relevant to certain curved space and nonlinear systems, particularly $H$-spaces and self-dual gauge fields. The reason for this is essentially twofold.

(1) When considered as an abstract metric space, the $\mathscr{I}^+$ of flat space–time is identical to that of any asymptotically flat space–time—all $\mathscr{I}^+$ are created equal.

(2) In the asymptotic region close to $\mathscr{I}^+$, the fields are small and thus, to some extent, a linear approximation becomes valid—at least for certain 'radiation' components.

Let us consider how this applies to $H$-spaces.

An $H$-space is a self-dual solution of the complexified, empty (source-free) Einstein equations which can be constructed from a function $\bar{\sigma}^0(u, \zeta, \bar{\zeta})$ of weight $(-2, -1)$. This function is usually interpreted as the asymptotic shear of some real solution. Since it is self-dual, the curvature of an $H$-space is completely described by a single Weyl spinor $\psi_{ABCD}$ which satisfies the Bianchi identities

$$\nabla^A_{A'}\psi_{ABCD} = 0,$$

where $\nabla^A_{A'}$ is now the 'curved' $H$-space spinor covariant derivative. We obviously cannot go into the details of the $H$-space construction here but, suffice it to say, any suitably well behaved function $\bar{\sigma}^0$ of weight $(-2, -1)$ can be used to generate an $H$-space which reduces to flat space–time when $\bar{\sigma}^0$ vanishes, or can be made to vanish by means of a supertranslation. In a sense, $\bar{\sigma}^0$ is a measure of the curvature of its associated $H$-space. For details of the $H$-space construction we refer the reader to Ko *et al* (1981).

If $\bar{\sigma}^0$ is small, one can form a perturbation expansion for an $H$-space by writing $\bar{\sigma}^0 = \tau S$ and expanding in powers of $\tau$. The first-order (linear) approximation can be shown to be a flat-space, spin-2 field which is identical to that generated by $S$ by the methods of § 3. The higher-order terms are much more complicated but—and this is the important point—they drop off faster than the first-order terms as $\mathscr{I}^+$ is approached if $S$ satisfies

$$S = O^*(u^{-\epsilon})$$

for large positive $u$. This is the condition for the first-order field to be asymptotically flat. Therefore, assuming that the perturbation expansion converges, an $H$-space is asymptotically flat if its first-order field is asymptotically flat and, in this case, both the $H$-space and its first-order field induce the same intrinsic structure on $\mathscr{I}^+$, i.e.

$$\psi_4^0 = -\ddot{\bar{\sigma}}^0 = \tau\phi_4^0, \qquad \psi_3^0 = \eth\dot{\bar{\sigma}}^0 = \tau\phi_3^0, \qquad \psi_2^0 = -\eth^2\bar{\sigma}^0 = \tau\phi_2^0.$$

From the point of view of $\mathscr{I}^+$, an asymptotically flat spin-2 field is identical to its associated $H$-space. Similarly, the condition

$$S = \pi(\zeta, \bar{\zeta}) + O^*(u^{-\epsilon}),$$

for large negative $u$, can be shown to imply that the $H$-space is asymptotically flat at $\mathscr{I}^-$. The relevance, if any, of the reality condition, $\bar{\eth}^2\pi = \text{real}$, to $H$-space theory is not, as yet, clear.

A similar argument also applies to self-dual gauge fields. By means of a construction formulated by Sparling and Newman (Newman 1980), almost any such gauge field can be generated from a matrix $\alpha_{ij}(u, \zeta, \bar{\zeta})$, of functions of weight $(-1, -1)$, where $i$ and $j$ refer to the representation of the gauge group. As in the $H$-space case, we can write $\alpha_{ij} = \tau S_{ij}$ and form a perturbation expansion.

To first order this yields several self-dual Maxwell fields which are identical to those generated by $S_{ij}$ by the methods of § 3. These first-order fields are again dominant near $\mathscr{I}^+$ if

$$S_{ij} = O^*(u^{-\epsilon}),$$

and hence (assuming that the perturbation expansion converges) the gauge field is asymptotically flat. In this case, both fields induce the same intrinsic structure on $\mathscr{I}^+$.

The results of this paper also appear to have some bearing on isolated (real) solutions of Einstein's equations, i.e. solutions which are supposed to describe well behaved isolated systems. In a previous paper (Ludvigsen 1981b) it was argued on certain physical grounds that an isolated solution should be asymptotically flat at $\mathscr{I}^+$, and that its asymptotic shear $\bar{\sigma}^0$ should satisfy

$$\bar{\sigma}^0 = O^*(u^{-\epsilon}) \qquad \text{for positive } u,$$

$$\bar{\sigma}^0 = \pi(\zeta, \bar{\zeta}) + O^*(u^{-\epsilon}) \qquad \text{for negative } u,$$

where $\bar{\eth}^2\pi$ is real. These conditions cause the associated spin-2 field, and hence the associated $H$-space, to be asymptotically flat. This leads one to expect that a defining feature of an isolated solution may be the asymptotic flatness of its associated $H$-space.

## Appendix

*Theorem.* If $\phi_{A\ldots Z}$ is an AF integer spin field with asymptotic component $\phi_n^0$ $(n = 2s)$, then

$$\phi_{A\ldots Z}(x) = -\oint \dot{\phi}_n^0(x^aL_a, \zeta, \bar{\zeta})O_A \ldots O_Z \, d\Omega. \tag{A1}$$

*Proof.* It is clear from the invariance properties of (A1) that it is sufficient to show that

$$\phi_{A\ldots Z}(0) = -\oint \phi_n^0(0, \zeta, \bar{\zeta}) O_A \ldots O_Z \, d\Omega \tag{A2}$$

where 0 is the origin point of our coordinate system. Since $O_A$ and $I_A$ satisfy $O_A I^A = 1$ and $\bar{\eth} O_A = -I_A$, $\phi_{A\ldots Z}(0)$ can be expanded in the form

$$\phi_{A\ldots Z}(0) = \frac{\bar{\eth}^n \beta}{n!} O_A \ldots O_Z + \frac{\bar{\eth}^{n-1}\beta}{(n-1)!} O_{(A} \ldots I_{Z)} + \ldots + \beta I_A \ldots I_Z \tag{A3}$$

where

$$\beta = \phi_{A\ldots Z} O^A \ldots O^Z. \tag{A4}$$

Note that $\beta$ is equal to $\phi_0$ evaluated at 0. Since $\oint d\Omega = 1$, equation (A3) yields

$$\phi_{A\ldots Z}(0) = \oint \phi_{A\ldots Z}(0) \, d\Omega = \frac{(n+1)}{n!} \oint \beta O_A \ldots O_Z \, d\Omega, \tag{A5}$$

where we have integrated by parts. We shall now use the field equations (2.22) and (2.23) to show that the right-hand side of (A2) is equal to the right-hand side of (A5).

By means of induction, one can easily show that the radial equations (2.23) imply that

$$\partial^n (r^n \phi_n)/\partial r^n = \bar{\eth}^n \phi_0, \tag{A6}$$

and hence

$$\lim_{r\to\infty} \frac{\partial^{n-1}}{\partial r^{n-1}} (r^n \phi_n) = \int_0^\infty \bar{\eth}^n \phi_0 \, dr. \tag{A7}$$

As the field is AF, we have

$$\phi_n = \phi_n^0 r^{-1} + O^*(r^{-(1+\epsilon)}),$$

and when this is substituted into (A7) we obtain

$$\int_0^\infty \bar{\eth}^n \phi_0 \, dr = \lim_{r\to\infty} \frac{\partial^{n-1}}{\partial r^{n-1}} (\phi_n^0 r^{n-1}) = (n-1)! \, \phi_n^0. \tag{A8}$$

Therefore

$$\oint \dot{\phi}_n^0 O_A \ldots O_Z \, d\Omega = \frac{1}{(n-1)!} \oint \int_0^\infty \bar{\eth}^n \dot{\phi}_0 O_A \ldots O_Z \, d\Omega \, dr. \tag{A9}$$

From equations (2.22) we have

$$\dot{\phi}_0 - \frac{\partial \phi_0}{\partial r} - \frac{\phi_0}{r} = -\frac{1}{r} \eth \phi_1$$

and hence

$$\bar{\eth}^n \dot{\phi}_0 = \frac{\partial}{\partial r} \bar{\eth}^n \phi_0 + \frac{\bar{\eth}^n \phi_0}{r} - \frac{1}{r} \bar{\eth}^n \eth \phi_1. \tag{A10}$$

By means of the commutation relation

$$\bar{\eth}\eth\eta - \eth\bar{\eth}\eta = 2s\eta,$$

one can show that

$$-\bar{\eth}^n \eth\phi_1 = n\bar{\eth}^{n-1}\phi_1 - \eth\bar{\eth}^n\phi_1,$$

and when this is substituted into (A10) we obtain

$$\bar{\eth}^n\dot{\phi}_0 = \frac{\partial}{\partial r}\bar{\eth}^n\phi_0 + \frac{\bar{\eth}^n\phi_0}{r} + n\frac{\bar{\eth}^{n-1}\phi_1}{r} - \eth\bar{\eth}^n\phi_1. \tag{A11}$$

From equations (2.23) we have

$$\frac{\partial}{\partial r}\phi_1 + \frac{n}{r}\phi_1 = -\frac{1}{r}\bar{\eth}\phi_0$$

and hence

$$\frac{n}{r}\bar{\eth}^{n-1}\phi_1 + \frac{1}{r}\bar{\eth}^n\phi_0 = \frac{-\partial}{\partial r}\bar{\eth}^{n-1}\phi_1. \tag{A12}$$

Substituting (A12) into (A11), we obtain

$$\bar{\eth}^n\dot{\phi}_0 = \frac{\partial}{\partial r}(\bar{\eth}^n\phi_0 - \bar{\eth}^{n-1}\phi_1) - \frac{\eth\bar{\eth}^n\phi_1}{r}. \tag{A13}$$

Finally, on substituting (A13) into (A9), integrating by parts and using equations (A4) and (A5) plus the fact that $\eth O_A = 0$, we obtain

$$\begin{aligned}\int \phi_n O_A \ldots O_Z \, \mathrm{d}\Omega &= \frac{+1}{(n-1)!}\left[\oint (\bar{\eth}^n\phi_0 - \bar{\eth}^{n-1}\phi_1) O_A \ldots O_Z \, \mathrm{d}\Omega\right]_0^\infty \\ &= \frac{-1}{(n-1)!}\oint\left(\bar{\eth}^n\beta - \frac{\bar{\eth}^n\beta}{n}\right) O_A \ldots O_Z \, \mathrm{d}\Omega \\ &= -\frac{(n+1)}{n!}\oint \bar{\eth}^n\beta O_A \ldots O_z \, \mathrm{d}\Omega \\ &= -\phi_{A\ldots Z}(0).\end{aligned}$$

## References

Ashtekar A and Hansen R O 1978 *J. Math. Phys.* **19** 1542
Couch W E and Newman E T 1972 *J. Math. Phys.* **13** 929
Ko M, Ludvigsen M, Newman E T and Tod K P 1981 *Phys. Rep.* **71** No 2
Ludvigsen M 1981a *J. Phys. A: Math. Gen.* **14** L27
—— 1981b *Gen. Rel. Grav.* **13** 135
Newman E T 1980 *Phys. Rev.* D **22** 3023
Newman E T and Penrose R 1966 *J. Math. Phys.* **7** 863
Penrose R 1965 *Proc. R. Soc.* A **284** 159
—— 1968 in *Battelle Recontres* ed C de Witt and J Wheeler (New York: Benjamin)